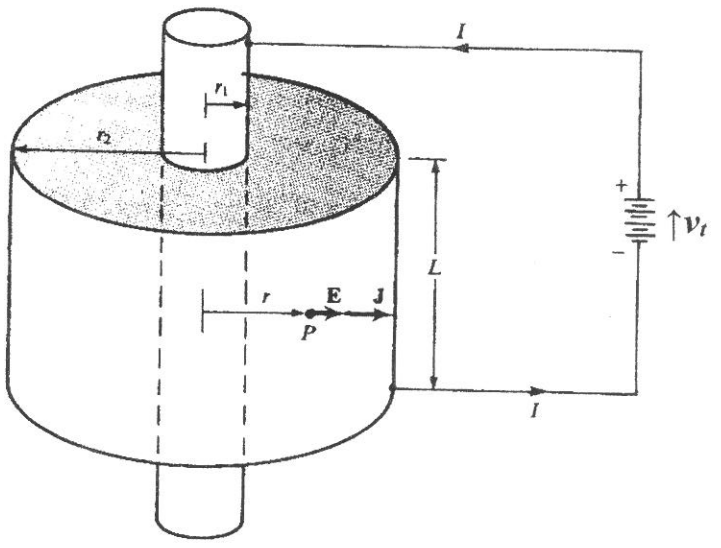
教育部 114 年公費留學考試試題 96

科目：普通物理 （全二頁，第一頁）

※可使用一般計算機(限僅具備＋、－、×、÷、% 、√、MR、MC、M+、M-運算功能)

※以中文或英文作答均可，評分基準相同。

一、 (25 points) Please refer to the figure below, a metal rod of radius r_1 is concentric with a metal cylindrical shell of radius r_2 and length L. The space between rod and cylinder is tightly packed with a high-resistance material of resistivity ρ. A battery having a terminal voltage v_t is connected as shown. Neglecting resistance of rod and cylinder, derive expressions for (a, 10 points) the total current I, (b, 10 points) the current density J, the electric field E at any point between rod and cylinder, and (c, 5 points) the resistance R between rod and cylinder.



二、 A man holds a ball of weight **w**=0.25 lb at rest in his hand. He then throws the ball vertically upward. In this process, his hand moves up 2 ft (feet) and the ball leaves his hand with an upward velocity of 48 ft/sec. What is the force **F** with which the man pushes on the ball? Note, you can use **g**=32 ft/s^2 in calculation. (25 points)

三、 (25 points) In a double-slit experiment, the wavelength λ of the light source is 405 nm, the slit separation d is 19.44 μm, and the slit width a is 4.050 μm. Consider the interference of the light from the two slits and also the diffraction of the light through each slit. (a, 10 points) How many bright interference fringes are within the central peak of the diffraction envelope? (b, 10 points) How many bright

(接下頁)

fringes are within either of the first side peaks of the diffraction envelope? (c, 5 points) Draw the interference plus the diffraction fringes for the center peak and the two side peaks next to the center peak.

四、(25 points) If a spherical solid non-conductor of radius ***R*** has a uniform charged ***Q*** distributed in the sphere. (a, 10 points) What is the electric field at a distance ***r*** outside the sphere, (b, 10 points) and the electric field at a distance ***r*** inside the sphere. (c, 5 points) Draw the relation of electric field as a function of distance ***r*** from the center. **(Please use blue or black color)**

教育部 114 年公費留學考試試題

科目：量子物理　　（全二頁，第一頁）

※以中文或英文作答均可，評分基準相同。

※可使用一般計算機(限僅具備＋、－、×、÷、%、√、MR、MC、M+、M-運算功能)

一、(總分 25 分)已知一維簡諧振子的哈密頓量（Hamiltonian）為 $H = \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2}$。其中 m 為粒子質量、k 為彈簧係數、$\hat{x}$ 為位移、$\hat{p}$ 為動量。

（一）引進上升算符 $\hat{a}^+$ 和下降算符 $\hat{a}$, 將哈密頓量轉化成

$$H = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$ 。(15 分)

（二）找出一維簡諧振子的基態能量 E_0，

i.e., $H|0\rangle = E_0|0\rangle$。(10 分)

二、(總分 25 分)根據包立算符（矩陣）

$$\hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \hat{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

我們可以定義，在三維笛卡兒（Cartesian）座標系統內的「*向量包立算符*（vector Pauli operator）」：

$$\vec{\hat{\sigma}} = \hat{\sigma}_x \cdot \overrightarrow{e_x} + \hat{\sigma}_y \cdot \overrightarrow{e_y} + \hat{\sigma}_z \cdot \overrightarrow{e_z} \ ,$$

這裏 $\overrightarrow{e_x}, \overrightarrow{e_y}, \overrightarrow{e_z}$ 是三維笛卡兒座標系統的基底向量。證明，給定這個三維笛卡兒座標系統內的任意一個長度歸一的向量 $\vec{a}$，它滿足

$$(\vec{\hat{\sigma}} \cdot \vec{a})(\vec{\hat{\sigma}} \cdot \vec{a}) = \hat{I}.$$

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科目：量子物理 （全二頁，第二頁）

三、(總分 30 分)我們採用 $|x\rangle$ 和 $|p\rangle$ 分別來標記位移 $\hat{x}$ 和動量 $\hat{p}$ 所對應的本徵量子態，i.e., $\hat{x}|x\rangle = x|x\rangle, \hat{p}|p\rangle = p|p\rangle$。

（一）利用平移算符 $\hat{T}(dx) = \hat{I} - i\frac{\hat{p}}{\hbar}\cdot dx$，這裏 $\hat{I}$ 是單位算符，$\hbar$ 是普朗克常數。推導出位移和動量空間的轉換： $\hat{x}|p\rangle = i\hbar\frac{\partial}{\partial p}|p\rangle$。(15 分)

（二）推導出位移表象和動量表象互為傅立葉轉換，i.e.,

$$\phi(x) = \langle x|\phi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dp\, e^{\frac{ixp}{\hbar}}\langle p|\phi\rangle$$。(15 分)

四、(總分 20 分)在斯特恩-革拉赫（Stern-Gerlach）實驗中，分別以下面兩種同位素所組成的原子束射入，我們可以觀察到幾條偏折的訊號（解釋你的答案）？

（一）射入的原子束是由基態 Na^{11} 原子組成。(10 分)

（二）射入的原子束是由基態 Na^{10} 原子組成。(10 分)

（試題隨試卷繳回）

教育部 114 年公費留學考試試題 98

科目：微積分 （全二頁，第一頁）

※可使用一般計算機(限僅具備＋、－、×、÷、% 、√、MR、MC、M+、M-運算功能)

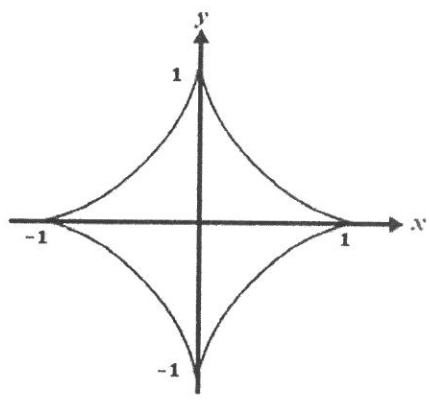
※以英文或中文作答皆可，評分基準相同，需標明題號，並詳述演算或推論過程。

1. (20 points)
   (a) Prove $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ by geometric methods and without using L'Hôpital's rule. (10 points)
   (b) Evaluate $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$ without using L'Hôpital's rule. (5 points)
   (c) Find the derivative of $f(x)=\sin x$ according to the definition of the derivative. (5 points)

2. (10 points) The region R is enclosed by the lines $y=x$, $y=2-x$ and $y=0$. Find the volume of the resulting solid by rotating R about: (a) $y=2$ (5 points) and (b) $x=3$. (5 points)

3. Find the limit $\lim_{x\to 2}\frac{x^3-2x^2-3x+6}{x-2}$ and prove your answer according to the $\varepsilon-\delta$ definition of the limit of a function. (15 points)

4. (15 points)
   (a) Find the length of the astroid defined as: $x=\cos^3\theta$, $y=\sin^3\theta$ where $0\le\theta\le 2\pi$, as shown in the figure below. (5 points)
   (b) Find the coordinates of the centroid of the first quadrant arc of the astroid defined in part (a). (10 points)



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教育部 114 年公費留學考試試題 98

科目：微積分 （全二頁，第二頁）

5. $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers that satisfies $a_n > 0$ for each n and $\sum_{n=1}^{\infty} a_n$ diverges. Show that $\sum_{n=1}^{\infty} \frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}$ converges, and find its sum. (15 points)

6. Find the area of the region enclosed by the curve $y = \sqrt{|x|}$ and the line $5y = x + 6$. (15 points)

7. Suppose $\int x^2 (\ln x)^2 \, dx = Ax^3 (\ln x)^2 + (Bx^3 + Cx^2 + Dx + E)(\ln x) + Fx^3 + Gx^2 + Hx + K$. Find the value of $A + B + F$. (10 points)

（試題隨試卷繳回）

教育部 114 年公費留學考試試題 99

科目：線性代數 （全二頁，第一頁）

※可使用一般計算機(限僅具備＋、－、×、÷、% 、√、MR、MC、M+、M-運算功能)

※以中文或英文作答均可，評分基準相同。

1. (30pts) Prove or disprove the following statements.

(a) Suppose A and B are 3×3 real matrices. If $AB = -BA$, then A or B is not invertible. (10pts)

(b) Let $f(t)$ be the characteristic polynomial of an $n \times n$ matrix A. Then

$$f(A) = \det(A - AI_n) = 0,$$

where I_n is the $n \times n$ identity matrix. (10pts)

(c) Let V be a complex vector space and $T : V \to V$ be linear. Then T is diagonalizable if and only if T^2 is diagonalizable. (10pts)

2. (20pts) Consider $V = P_2(\mathbb{R})$ be the collection of all real polynomials of degree less than or equal to 2. Define an inner product on V by

$$< f, g >= \int_0^1 f(x)g(x)dx$$

for all f(x), g(x) in V.

(a) Verify $<,>$ is really an inner product on V. (5 pts)

(b) Find an orthonormal basis $\{p_0(x), p_1(x), p_2(x)\}$ for V such that the degree of $p_i(x)$ is i for $i = 0, 1, 2$. (10 pts)

(c) Suppose $\{q_0(x), q_1(x), q_2(x)\}$ is also an orthonormal basis for V such that the degree of $q_i(x)$ is i for $i = 0, 1, 2$. Show that $q_i(x) = p_i(x)$ or $q_i(x) = -p_i(x)$ for $i = 0, 1, 2$, where $\{p_0(x), p_1(x), p_2(x)\}$ is the basis in (b). (5 pts)

3. (15 pts) Let V and W be finite-dimensional inner product spaces, and let $T : V \to W$ be linear. Show that if T is onto, then TT^* is invertible.

（接下頁）

科目：線性代數 （全二頁，第二頁）

4. (15 pts) Let V be a finite dimensional complex vector space and $T: V \to V$ be linear. Suppose

$$p(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0$$

is a polynomial with complex coefficients $c_i, i = 0,1, \ldots, n$. Let

$$U = p(T) = c_n T^n + c_{n-1} T^{n-1} + \cdots + c_0 I.$$

Show that μ is an eigenvalue of U if and only if $\mu = p(\lambda)$ for some eigenvalue λ of T.

5. (20 pts) Suppose $f: \mathbb{R}^3 \to \mathbb{R}^3$ be a function that satisfies $\|f(x) - f(y)\| = \|x - y\|$ for any vectors x, y in $\mathbb{R}^3$. Show that there exist a unique vector v and an orthogonal matrix U such that

$$f(x) = v + Ux$$

for any vector x in $\mathbb{R}^3$.

（試題隨試卷繳回）

教育部 114 年公費留學考試試題 100

科目：數理統計與機率 （全二頁，第一頁）

※可使用一般計算機(限僅具備＋、－、×、÷、% 、√、MR、MC、M+、M-運算功能)

※以中文或英文作答均可，評分基準相同。

一、 (總分 20 分) Consider the Gamma distribution with parameters $\alpha, \beta > 0$. The probability density function is given by

$$f(x;\ \alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x), \qquad x > 0.$$

(a) Derive the moment generating function (MGF) of the Gamma distribution. (7 分)

(b) Use the MGF obtained in (a) to derive the mean and variance of the Gamma distribution. (6 分)

(c) Let $X \sim \text{Gamma}(\alpha_1, \beta)$ and $Y \sim \text{Gamma}(\alpha_2, \beta)$ be two independent random variables. Show that the sum of X and Y is also a Gamma random variable. (7 分)

二、 Let X be a random variable with moment generating function $M_X(t)$, $-h < t < h$. Prove that $P(X \geq a) \leq e^{-at} M_X(t)$, $0 < t < h$. (10 分)

三、 Let $X_1, \ldots, X_n$ be i.i.d random variables from $N(\mu, \sigma^2)$ and

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2, \quad \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Show that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. (10 分)

四、 (總分 20 分) Let $X_1, \ldots, X_n$ be i.i.d from Unif(0,1) and $Y = min(X_1, \ldots, X_n)$.

(a) Derive the probability density function of Y. (10 分)

(b) Find the asymptotic distribution of nY. (10 分)

（接下頁）

五、 (總分 20 分) Let $X_1, \ldots, X_n$ be i.i.d random variables from $N(\mu, \sigma^2)$ where σ^2 is known and

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

(a) Show that $Z = \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$ is a pivotal quantity for μ. (10 分)

(b) Derive a $(1-\alpha)100\%$ confidence interval for μ using the pivotal quantity in (a). (10 分)

六、 (總分 20 分) Let $X_1, \ldots, X_n$ be i.i.d random variables from Exp(λ), where the probability density function is given by

$$f(x;\lambda) = \lambda^{-1} exp\left(-\frac{x}{\lambda}\right), \quad x > 0.$$

Consider testing the null hypothesis $H_0: \lambda = \lambda_0$ against the alternative hypothesis $H_1: \lambda \neq \lambda_0$.

(a) Derive the likelihood ratio test statistic. (10 分)

(b) Derive the asymptotic distribution of the likelihood ratio test statistic under the null hypothesis. (10 分)

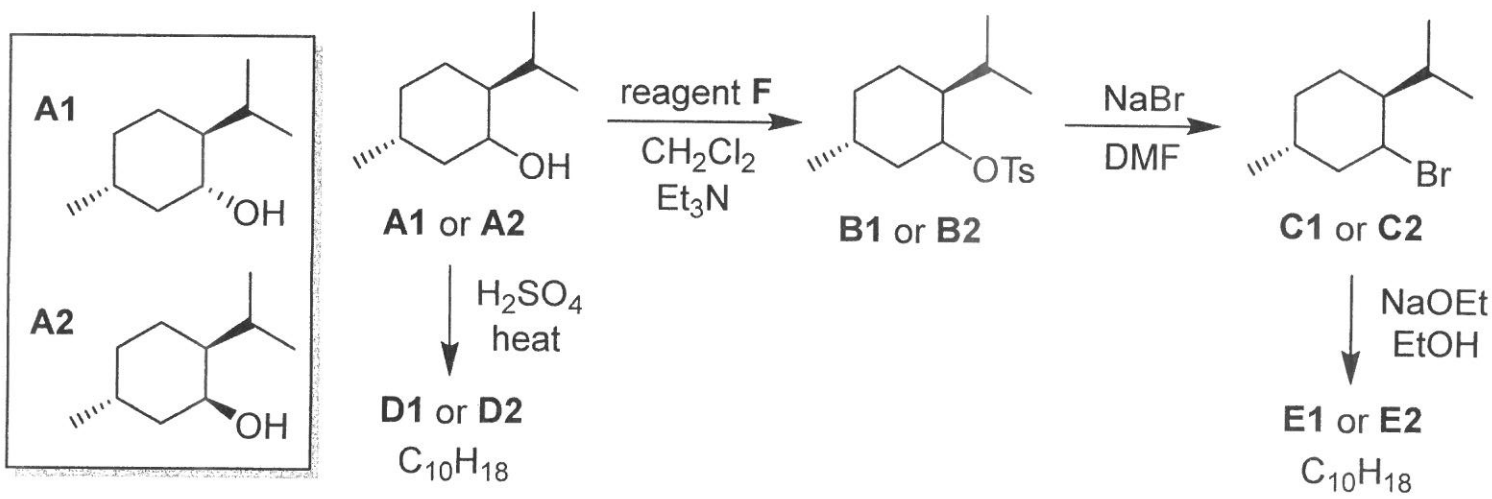
教育部 114 年公費留學考試試題

科目：有機化學 （全六頁，第一頁）

※以中文或英文作答均可，評分基準相同。

※可使用一般計算機(限僅具備＋、－、×、÷、% 、√、MR、MC、M+、M-運算功能)

1. (14pts) Compound **A1** undergoes a series of transformations to produce **B1**, **C1**, **D1**, and **E1** ($C_{10}H_{18}$) as the major products in the following scheme. The same scheme can also be applied to compound **A2**, **B2**, **C2**, **D2**, and **E2** ($C_{10}H_{18}$). (Hint: stereochemistry is very important here.)



(a) What is the chemical structure of reagent **F**? (2pts)

(b) Predict the structures of product **C1** and **C2**. (2pts)

(c) Predict the structures of the major product **D1** and **D2**. Compare the rates of reaction **A1** to **D1** and reaction **A2** to **D2**. Explain briefly. (5pts)

(d) Predict the structures of the major product **E1** and **E2**. Compare the rates of reaction **C1** to **E1** and reaction **C2** to **E2**. Explain briefly. (5pts)

2. (10pts) Four alkyl bromides were subjected to hydrolysis in a mixture of ethanol (80%) and water (20%) at 55°C, the rates of the reaction showed the following order:

（接下頁）

$(CH_3)_3CBr > CH_3Br > CH_3CH_2Br > (CH_3)_2CHBr$

(a) Provide an explanation for this order of reactivity. (4pts)

(b) Use reaction energy diagrams to explain the reaction rates difference between $(CH_3)_3CBr$ and CH_3Br. Draw key transition state(s), inter-mediate(s) and label key free energy items in a relatively comparable manner. (6pts)

3. (9pts) When furan (C_4H_4O) and maleimide ($C_4H_3NO_2$) undergo a Diels-Alder reaction at 25°C, the major product is the adduct **A**. When the reaction is carried out at 90 °C, however, the major product is another adduct **B**. The adduct **A** and **B** are different isomers.

(a) Use the molecular orbital (MO) theory to describe this Diels-Alder reaction and predict two possible isomers. (4pts)

(b) Use the reaction energy diagram to explain how different temperatures affect the product distribution using. Predict the structures of product **A** and **B** under different temperatures. (5pts)

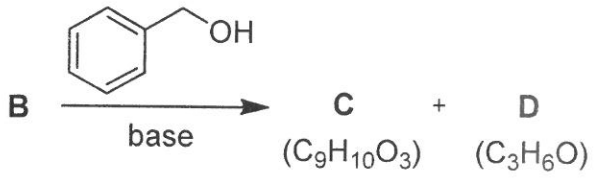
4. (12pts) Both 1,2-diols or α-hydroxy carboxylic acids, as shown in the following examples, can be protected by acetone (C_3H_6O).

（接下頁）

(a) Provide structures of **A** and **B** in the corresponding boxes. (2pts)

(b) When acetone is replaced by 2,2-dimethoxypropane ($C_5H_{12}O_2$), the formation of **A** gave a better yield even under milder acidic conditions. Provide a mechanism and explanation of these outcomes. (4pts)

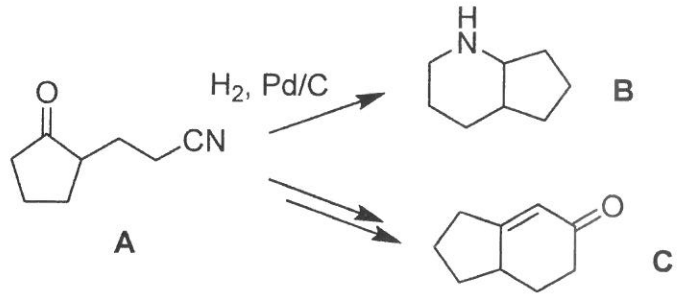
Interestingly, under basic conditions, compound **B** is not stable and can undergo esterification to afford compound **C** and **D** in one step as shown below.



(c) What are product **C** and **D**? (3pts)

(d) Provide a mechanism to explain this transformation from **B** to **C** and **D**. (3pts)

5. (7pts) Compound **B** and **C** can be prepared from **A** through the following synthetic scheme.



(a) Provide a mechanism for the transformation of **B** from **A**. (Note: you do not need to give the mechanism of H_2+Pd/C) (3pts)

(b) Provide a synthetic scheme to prepare **C** from **A**. (4pts)

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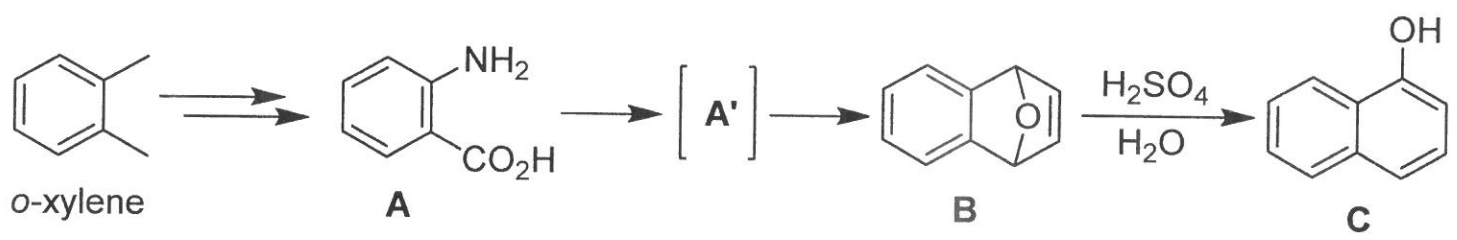
6. (12pts) Compound **C** can be prepared through the following synthetic scheme.



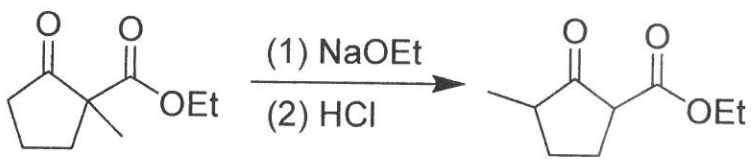
o-xylene **A** **B** **C**

(a) Provide a mechanism for the transformation from **B** to **C**. What is the driving force of this transformation? (4pts)

(b) Compound **A** is activated into the transient intermediate **A'** during the transformation from **A** to **B**. Complete this transformation and give a mechanism to explain how **A** is activated into **A'**. (4pts)

(c) Provide a synthetic scheme and to prepare **A** from *o*-xylene. Give a mechanism to explain how one carbon is lost in this transformation. (4pts)

7. (8pts) The following transformation from **A** to **B** occurs in given conditions.



A **B**

(a) Suggest a reasonable mechanism and explain its driving force. (4pts)

(b) Outline a preparation scheme of compound **A** from **cyclohexene**. (Note: methyl group is from CH_3I) (4pts)

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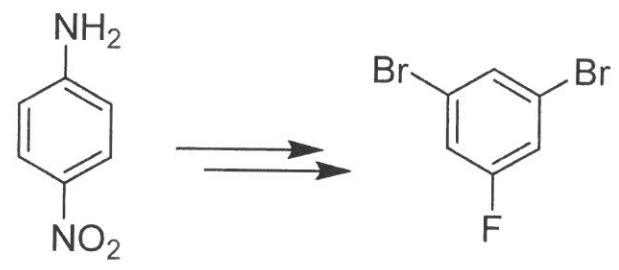
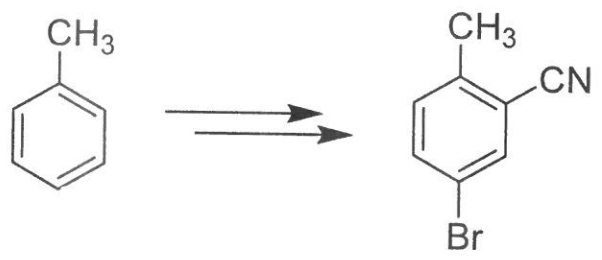
8. (8pts) Provide synthetic schemes with proper reagents for the following transformations. (4pts each)

(a) 

(b) 

9. (20pts) **DEHP** ($C_{24}H_{38}O_4$) is a common plasticizer that can be added to plastics to increase plasticity. However, it is also an environmental hormone that can interfere with endocrine function. Therefore, its legal usage must be strictly controlled, and it must not be misused as a food additive. In industrial synthesis, **A** ($C_8H_{18}O$) and **B** ($C_8H_6O_4$) are combined under condition **C** to produce **DEHP** and the byproduct H_2O.

A ($C_8H_{18}O$) can react with sodium metal to produce hydrogen. **A** can also react with cold $KMnO_4$ solution to form **D** ($C_8H_{16}O_2$). **D** has *only* one chiral center. If **D** undergoes decarboxylation, eliminating CO_2, it will form *n*-heptane (C_7H_{16}).

(a) What functional group does **A** possess, and determine the degree of the carbon atom to which this functional group is attached? (2pts)

(b) Provide all possible structures of **D**. (2pts)

D can be prepared from **E** ($C_7H_{15}Br$) with CO_2. When treated with sodium ethoxide, **E** produces a mixture of alkene **F**s (C_7H_{14}), which all exhibit *cis-trans* isomerism.

(c) Provide the chemical structure of **E** and **F**. (4pts)

(d) Based on all information from (a) to (c), deduce the chemical structure of **A**. (2pts)

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科目：有機化學 （全六頁，第六頁）

In industry, the preparation of **A** is more straightforward and can be generated *solely* from an open-chain (acyclic) four-carbon compound **G** (C_4H_8O).

(e) Provide a synthetic scheme to prepare **A** from **G**. (3pts)

B ($C_8H_6O_4$) is an acidic substance with two functional groups that have measurable pKa values of 2.89 and 5.51, respectively. **B** does not decolorize bromine water or neutral potassium permanganate solution. **B** can form an acid anhydride.

(f) What is the chemical structure of **B**? (2pts)

(g) Provide the condition **C** and the structure of **DEHP**, along with the correctly balanced reaction equation for the preparation of **DEHP** from **A** and **B**. (2pts)

(h) Draw all possible stereoisomers of **DEHP**. Which of these isomers are chiral? (3pts)

(試題隨試卷繳回)

教育部 114 年公費留學考試試題　102

科目：分析化學　　**（全一頁）**

※以中文或英文作答均可，評分基準相同。

※可使用**一般**計算機(限僅具備＋、－、×、÷、%、√、MR、MC、M+、M-運算功能)

一、(總分 25 分)現代光譜儀器中，偵測器（transducer）扮演關鍵角色。無論是在環境監測中的微量氣體檢測，或是醫學影像中對生物分子的紅外光訊號分析，都必須選擇合適的感測方式。偵測器大致可分為光子型（量子型）與熱感應型兩大類。

(一) 請比較光子型（量子型）轉換器與熱感應型轉換器在原理、響應速度、靈敏度、適用光譜範圍上的異同。(15 分)

(二) 為何一般常見的光子型轉換器無法用來偵測紅外線輻射？(10 分)

二、(總分 25 分)玻璃電極作為酸鹼度計，在設計中，利用接界電位（liquid junction potential, E_j）以及邊界電位（boundary potential, E_b）決定分析樣品的酸鹼度值（pH）。使用玻璃電極測定 pH，已知參考電極電位為E_{ref}=0.200 V，邊界電位遵循能斯特方程式。若內部參考液 pH=7.00，測得全電池電位 E_{cell}=0.415 V（25 °C）：

(一) 寫出並說明玻璃膜能斯特方程式的來源(由電化學勢/能斯特關係出發)；說明液接界電位如何導向邊界電位的形式與物理意義。(15 分)

(二) 利用全電池電位計算外部溶液的 pH。(10 分)

三、(總分 25 分)

(一) 在分析儀器中，電子訊號的處理與濾波是確保量測精度的重要步驟。請定義高通濾波器（high-pass filter）與低通濾波器（low-pass filter），並繪製其等效 RC 電路，寫出對應的轉移函數與截止頻率表達式。(15 分)

(二) 在實際應用中，不同頻率範圍的雜訊干擾對量測精度影響不同。請說明：

1. 何種雜訊干擾可透過高通濾波器有效抑制？(5 分)
2. 何種雜訊干擾可透過低通濾波器有效抑制？(5 分)

四、(總分 25 分)在原子吸收光譜儀（Atomic Absorption Spectroscopy, AAS）的設計與操作中，光源（如中空陰極燈, Hollow Cathode Lamp, HCL）發出的特徵輻射除了被待測元素吸收外，樣品基質也可能同時產生散射、背景輻射。請說明：為何在 AAS 儀器設計中必須使用「光源調變（source modulation）」技術，討論其在儀器光路、雜訊抑制與訊號解析中的角色(15 分)，並簡述所對應的電子檢測方式。(10 分)

（試題隨試卷繳回）