教育部112年公費留學考試試題 96

科目：普通物理 （全二頁，第一頁）

※可使用一般計算機(限僅具備＋、－、×、÷、%、√、MR、MC、M+、M-運算功能)

※以中文或英文作答均可，評分基準相同。

※若需繪圖請以藍色或黑色原子筆做圖。

一、(20%) In Fig.1, three interconnected blocks are pulled to the right on a horizontal frictionless table by a force of magnitude T_3 = 65.0 *N*. Given that m_1 = 12.0 kg, m_2 = 24.0 kg, and m_3 = 31.0 kg, calculate (a) the magnitude of the system's acceleration, (10%) (b) the tension T_1 (10%)

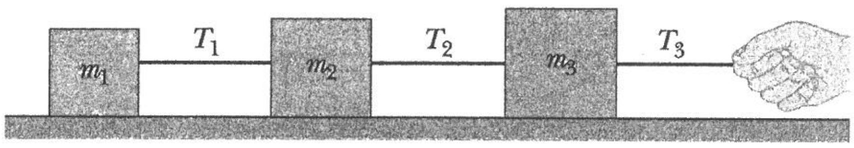


Fig.1

二、(20%) Fig. 2 illustrates a *conical pendulum,* wherein the bob (the small object at the lower end of the cord) moves in a horizontal circle at constant speed. (As the bob rotates, the cord sweeps out a cone shape) The bob has a mass of *m*, the string has length *L* and negligible mass, and the bob follows a circular path of circumference $2\pi r$. Determine (a) the tension in the string (10%) and (b) the period of the motion? (10%)

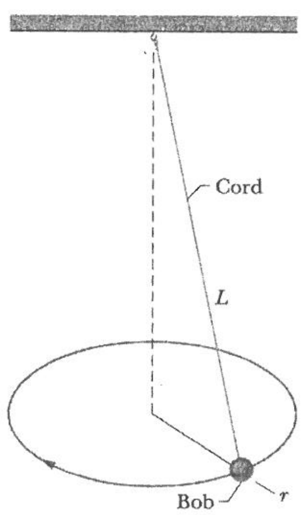


Fig.2

(接下頁)

三、(20%) A simple harmonic oscillator (S.H.O.) consists of a mass m coupled to a spring with Hooke's constant k. The equation of motion is given by $F = m\ dv/dt = -k\ x$, where v is the velocity of the object. (a) Prove that $\frac{1}{2}mv^2 + \frac{1}{2}kx^2$ is constant.(10%) (b) Please sketch a diagram in the phase space (x, p) for the aforementioned case. Where the linear momentum $p = mv$ (10%)

四、(20%) The *Carnot Cycle* is an ideal thermodynamic cycle. Considering the given P-V diagram, there's a working machine with a cycle A→B→C→D→A and a chamber containing 10 mol of ideal gas. The AB and CD curves represent isothermal processes, while the BC and DA curves represent adiabatic processes. Assuming $T_2 = 2T_1$ and $V_C = 8V_A$, and given the gas constant R, please determine

(a) the total work done, W_{cycle}, (10%) and

(b) the thermal efficiency, η (10%).

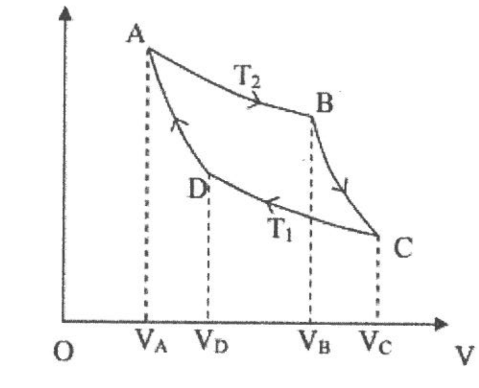


Fig.3

五、(20%) Light with a wavelength of 200 nm falls on an aluminum surface with a work function of 4.2 eV. (a) Determine the stopping potential. (10%)

(b) Calculate the cutoff wavelength. (10%)

（試題隨試卷繳回）

教育部112年公費留學考試試題 97

科目：量子物理　　　　　　　　　　　　　　　　（全二頁，第一頁）

※可使用一般計算機(限僅具備＋、－、×、÷、%、√、MR、MC、M+、M-運算功能)

一、(總分30分)一質量為m的粒子被束縛在一維$0 \le x \le a$的無限高位能井內。

其初始波函數為$\Psi(x,0) = \begin{cases} Bx, & 0 \le x \le a/2 \\ B(a-x), & a/2 \le x \le a \end{cases}$，求解：

(一)歸一常數B。(10分)

(二)$\Psi(x,t)$。(10分)

(三)發現此粒子處於無限高位能井基態(ground state)的機率。(10分)

二、(總分30分)一質量為m的粒子被束縛在一維位勢$V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2 + \beta\hat{x}$中，其中$\omega$為自然角頻率，$\hat{x}$為位置算符，$\beta$為實數微擾常數。在一維簡諧振子的系統中可定義$|n\rangle$是量子數為$n$的本徵態向量，下降算符$\hat{a} = i\sqrt{\frac{1}{2m\hbar\omega}}\hat{p} + \sqrt{\frac{m\omega}{2\hbar}}\hat{x}$，上升算符$\hat{a}^\dagger = -i\sqrt{\frac{1}{2m\hbar\omega}}\hat{p} + \sqrt{\frac{m\omega}{2\hbar}}\hat{x}$，其中$\hat{p}$是動量算符，$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$，$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$。

(一)試用$|0\rangle$和$|1\rangle$兩個能階，建構此粒子在$V(\hat{x})$中的Hamiltonian 2×2矩陣。(10分)

求解粒子在$V(\hat{x})$位勢中的基態與第一激發態的

(二)本徵能量(eigenenergy)。(10分)

(三)本徵態向量(eigenstate)。(10分)

(接下頁)

三、 承上題，下降算符$\hat{a}$的本徵態稱為同調態(coherent state)$|\alpha\rangle$，即$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$。若$|\alpha\rangle = \sum_{n=0}^{\infty} C_n|n\rangle$，且已知$C_0$，試求出$C_{n\geq 1}$。(20 分)

四、 用推廣的 Ehrenfest 定理$\frac{d}{dt}\langle\hat{O}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{O}]\rangle + \langle\frac{\partial\hat{O}}{\partial t}\rangle$求解單電子在指向為直角座標系$\hat{z}$方向的定磁場$\vec{B} = B\hat{z}$中的 Larmor 進動(Larmor precession)，其中$\hat{O}$為任意算符，$\hat{H}$為 Hamiltonian 算符，t為時間。單電子的磁矩算符$\hat{\mu}$和自旋角動量算符$\hat{s}$的關係為$\hat{\mu} = \gamma\hat{s}$，其中$\gamma$是磁旋比常數，請計算出期望值$\langle\hat{s}\rangle$在直角座標系的三個分量對時間的關係。(20 分)

教育部 112 年公費留學考試試題　98

科目：微積分　（全二頁，第一頁）

※可使用一般計算機(限僅具備＋、－、×、÷、% 、√、MR、MC、M+、M-運算功能)

※以中文或英文作答均可，評分標準相同。

※請詳述演算或推論過程，並標明題號。

1. (15 points) For each $n \geq 1$, define $a_n = \frac{1}{n}\sum_{k=1}^{n}\cos\frac{\pi}{k}$. Prove that $\{a_n\}$ is convergent and evaluate its limit.

2. (15 points) Prove or disprove the following statement: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty}\sin a_n$ is convergent.

3. (15 points) Determine whether the improper integral $\int_0^{\infty}\sin\left(e^x\right)dx$ is convergent or divergent.

4. (15 points) Let $a_n = \int_0^1 \left(1-x^2\right)^n dx$ for $n \geq 0$. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$.

5. (10 points) Find the length of the arc of the curve $y = \frac{x^2}{2} - \frac{1}{4}\ln x$ from point $P = \left(1, \frac{1}{2}\right)$ to point $Q\left(5, \frac{50-\ln 5}{4}\right)$.

6. (10 points) Let $\mathbf{u}: I \to \mathbb{R}^3$ be a twice differentiable vector function defined on an open interval I. Suppose that $\mathbf{u}$ satisfies the second order ordinary differential equation $\mathbf{u}''(t) = c\mathbf{u}$, where c is a nonzero constant. Prove that there exists a vector $\mathbf{h}$ in $\mathbb{R}^3$ such that $\mathbf{u}(t)\times\mathbf{u}'(t) = \mathbf{h}$ for all $t \in I$. Here $\times$ is the cross product between vectors in $\mathbb{R}^3$.

（接下頁）

科目：微積分

7. (10 points) A function $f:\mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}$ is a radial function if there exists a function $g:(0,\infty) \to \mathbb{R}$ such that $f(x,y,z) = g(r)$ where $r = \sqrt{x^2+y^2+z^2}$. Find all twice differentiable radial functions f defined on $\mathbb{R}^3 \setminus \{(0,0,0)\}$ such that

$f_{xx} + f_{yy} + f_{zz} = 0$.

8. (10 points) Evaluate the double integral $\iint_D \sin\left(\sqrt{x^2+y^2}\right) dA$, where D is the closed unit disk centered at the origin.

教育部112年公費留學考試試題 99

科目：線性代數 （全二頁，第一頁）

※可使用一般計算機(限僅具備＋、－、×、÷、% 、√、MR、MC、M+、M-運算功能)

※以中文或英文作答均可，評分基準相同。

Notation: $\mathbb{R}$ is the set of real numbers. For each positive integer n, denote by $M_n(\mathbb{R})$ the set of all n-by-n matrices with entries in $\mathbb{R}$. Let 0_n be the zero matrix in $M_n(\mathbb{R})$, and I_n be the identity matrix in $M_n(\mathbb{R})$.

1. (15pts) Let V be a vector space of dimension 6 over $\mathbb{R}$, and $\{e_1, \dots, e_6\}$ be a basis of V. Let $T, T': V \longrightarrow V$ be two linear transformations satisfying that for $1 \le i \le 6$,

$$T(e_i) = e_i - e_{i'}, \quad \text{where} \quad i' = \begin{cases} i+4, & \text{if } i \le 2, \\ i-2, & \text{if } i > 2, \end{cases}$$

and

$$T'(e_i) = \sum_{j=1}^{6} a_{ji} e_j, \qquad \text{where} \quad a_{ij} = \begin{cases} 2^{i+j-2}, & \text{if } i+j < 8, \\ 2^{i+j-8}, & \text{if } i+j \ge 8. \end{cases}$$

Put $L = T \circ T': V \longrightarrow V$. Find a basis of the null space of L and a basis of the range of L (expressing as linear combinations of $e_1, \dots, e_6$).

2. (15pts)Let

$$A = \begin{bmatrix} 4 & 7 & 5 \\ -3 & -6 & -5 \\ 3 & 7 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & -6 & -4 \\ 3 & 7 & 6 \\ -3 & -9 & -8 \end{bmatrix}.$$

(1)(5pts) Verify that $AB = BA$.

(2)(10pts) Find an invertible matrix $U \in M_3(\mathbb{R})$ such that $U^{-1}AU$ and $U^{-1}BU$ are both diagonal matrices.

3. (20pts) Let

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Determine whether there exists an invertible matrix $U \in M_4(\mathbb{R})$ such that $U^{-1}BU = A$. If such a matrix U exists, find one.

（接下頁）

4. (15pts) Prove or disprove that there exists a matrix $A \in M_3(\mathbb{R})$ satisfying that

$$A^4 + A^3 + A^2 + A + I_3 = 0_3.$$

5. (15pts) Let V be a finite dimensional vector space over $\mathbb{R}$, and $T, T': V \longrightarrow V$ are two linear transformations. Suppose that a real number λ is an eigenvalue of $T \circ T'$. Prove or disprove that λ is also an eigenvalue of $T' \circ T$.

6. (20pts) Let V be a vector space of dimension n over $\mathbb{R}$, and let $T: V \longrightarrow V$ be a linear transformation. Suppose $n \geq 2$. Show that there exists a 2-dimensional subspace W of V which is T-invariant, i.e. $T(W) \subseteq W$.

教育部112年公費留學考試試題 100

科目：數理統計與機率 **（全二頁，第一頁）**

※可使用一般計算機(限僅具備＋、－、×、÷、% 、√、MR、MC、M+、M-運算功能)

※以中文或英文作答均可，評分基準相同。

一、(總分15分)Let X and Y be i.i.d. standard normal random variables, and let (R, Θ) be the polar coordinates of the point (X, Y). In other words, R is the length of the vector (X, Y), and Θ is the angle between the line segment from $(0,0)$ and (X, Y) and the x-axis (where $0 \leq \Theta < 2\pi$).

A. (3分) Find the joint probability density function of (R, Θ).

B. (2分) Are R and Θ independent?

C. (10分) Find the marginal probability density function of R, and compute $E[R]$.

二、(總分15分)Let X and Y be i.i.d. exponential random variables with parameter 1. Define $Z = X - Y$. Show that the probability density function of Z is given by $f(z) = \frac{1}{2}e^{-|z|}$ for $-\infty < z < \infty$.

三、(總分20分)Let $X_1, X_2, \ldots$ be i.i.d. random variables with mean μ, and let N be a positive integer-valued random variable with mean λ that is independent of the sequence (X_n). For $n \geq 1$, define $S_n = X_1 + \cdots + X_n$. Find $E[S_N]$.

四、(總分20分)

A. (10分)Prove the Markov inequality: if X is a nonnegative random variable and $a > 0$ is a positive real number, then

$$P(X \geq a) \leq \frac{E[X]}{a}.$$

(接下頁)

B. (5 分)Using the Markov inequality, deduce the Chebyshev inequality: if Y is a random variable with finite mean and variance then for any $\varepsilon > 0$

$$P(|Y - E[Y]| \geq \varepsilon) \leq \frac{\operatorname{Var}(Y)}{\varepsilon^2}.$$

C. (5 分)Let (X_n) be a sequence of i.i.d. random variables with finite mean and variance. Using the Chebyshev inequality, prove the weak law of large numbers:

$$\frac{X_1 + \cdots + X_n}{n} \longrightarrow E[X_1] \quad \text{in probability as } n \to \infty.$$

五、(總分 15 分)Suppose that X is a geometric random variable with parameter θ. That is, X has probability mass function $f(x) = \theta(1-\theta)^x$ for $x = 0,1,2,\ldots.$ Suppose that the prior distribution of θ in uniform in $[0,1]$. Determine the posterior distribution of θ given $X = x$.

六、(總分 15 分)

A. (5 分)Write down the definition of an unbiased estimator of a function $g(\theta)$ of a parameter θ.

B. (10 分)Let $X = (X_1, \ldots, X_n)$ (where $n \geq 2$) be a random sample from a distribution that depends on a parameter θ, and define $g(\theta) = \operatorname{Var}_\theta(X_1)$.

Show that

$$\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{j=1}^{n} X_j\right)$$

is an unbiased estimator of $g(\theta)$.

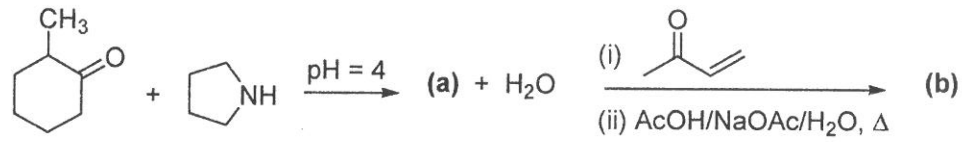
教育部 112 年公費留學考試試題 101

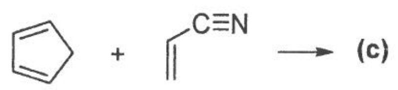
科目：有機化學 （全三頁，第一頁）

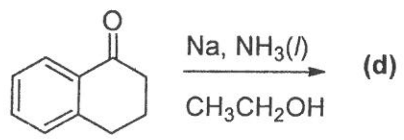
※可使用一般計算機(限僅具備＋、－、×、÷、% 、√、MR、MC、M+、M-運算功能)

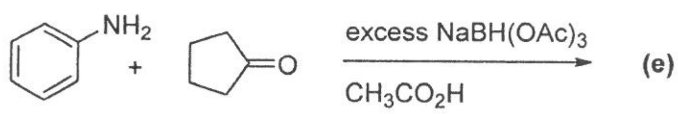
※以中文或英文作答皆可，評分基準相同。

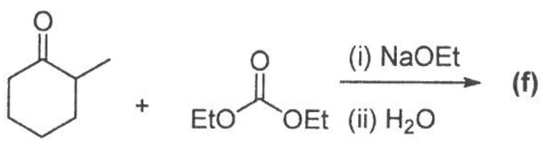
1. Predict the major product of the following reactions. Be sure to indicate stereochemistry where applicable. (30%, 3% each)

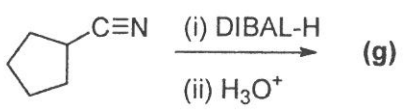


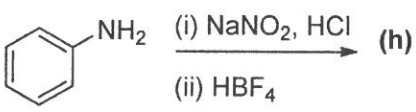


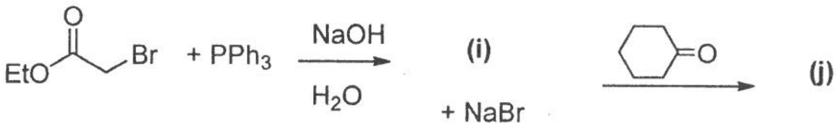






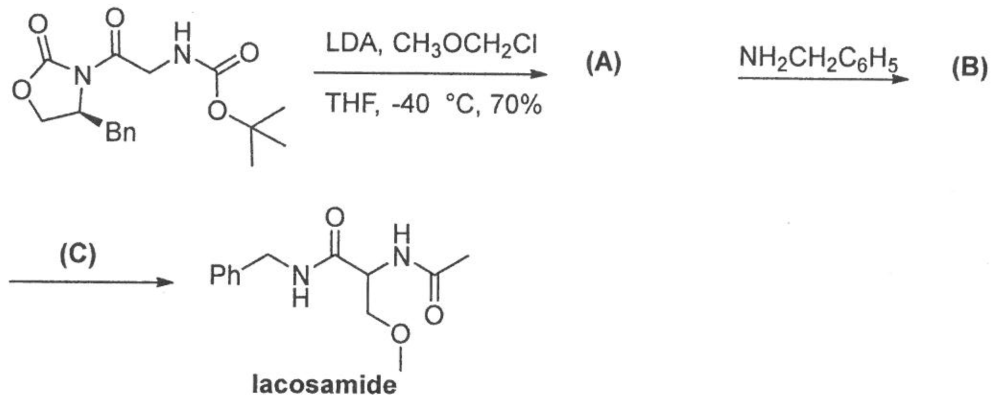




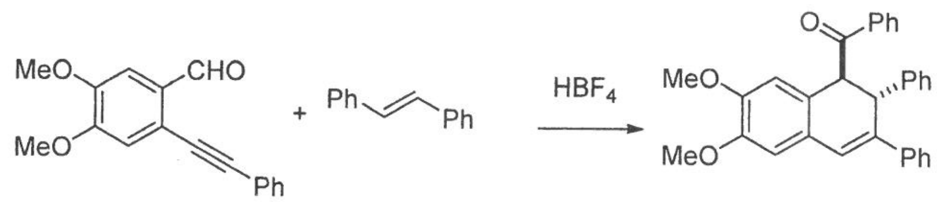


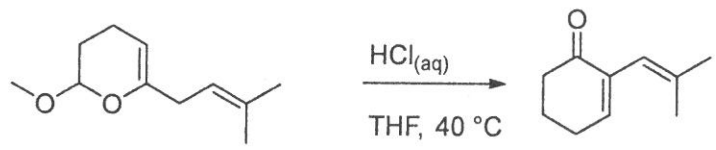
（接下頁）

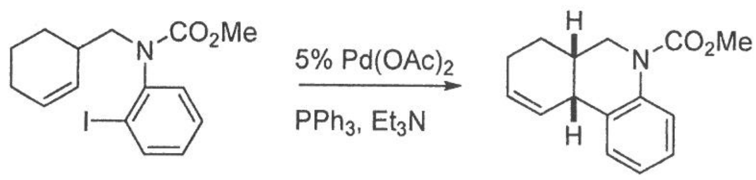
2. Lacosamide is a medication used for the treatment of partial-onset seizures, and one of its syntheses is shown below.
   (a) Provide the synthetic intermediates (A), (B) and the reagents (C) required to complete the synthesis. (12%)
   (b) According to this synthesis, what is the stereochemistry of lacosamide? (4%)

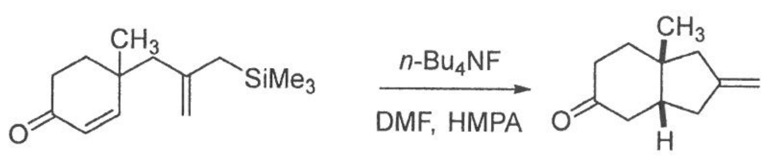


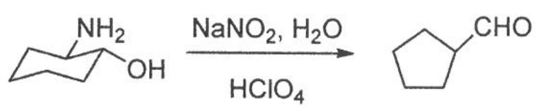
3. Propose a step-by-step mechanism for each of the following reactions. (25%, 5% each)

(a)


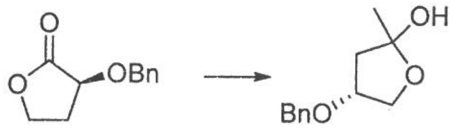
(b)


(c)


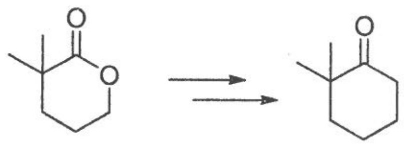
(d)


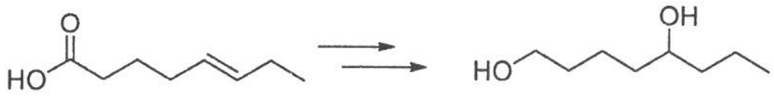
(e)


4. Propose a synthesis to accomplish the following transformations. Give the structures of intermediated and provide the required reagents. (24%, 6% each)

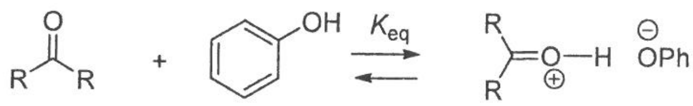
(a) 

(b) 

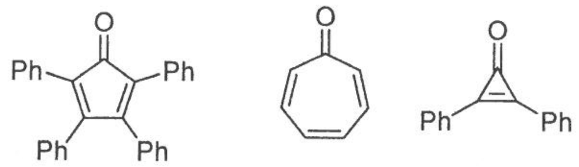
(c) 

(d) 

5. Explain the observed equilibrium constants (K_{eq}) of the following ketones **A-C** with phenol. (5%)



Ketones:



A	B	C
K_{eq} = 6.2	K_{eq} = 31.2	K_{eq} = 83.2

教育部112年公費留學考試試題 102

科目：分析化學 （全一頁）

※可使用一般計算機(限僅具備＋、－、×、÷、% 、√、MR、MC、M+、M-運算功能)

一、(總分25分)自2019年底開始全球爆發COVID-19大流行，造成人類生活的大改變，「快篩」成為日常，快篩試劑(常見為抗體檢測型)的開發為分析化學的重要應用，「偽陰性(false negative)」、「偽陽性(false positive)」、「普篩」等名詞常常見諸於新聞媒體，請依照常見快篩試劑的分析原理討論何謂「偽陰性」、「偽陽性」以及屬於分析化學何種誤差(error)來源，(15分)利用重複篩檢的方式是否能改善分析的結果(降低誤差)，請根據數據統計的概念說明並解釋其原因。(10分)

二、(總分25分)玻璃電極廣用於酸鹼度計(pH meter)的開發，請繪製利用玻璃電極所製作的酸鹼度計的設計結構(說明組成電極各自用途以及內部電解質成分)，(10分)並說明液體接界電位(liquid junction potential)以及邊界電位(boundary potential)的來源與區別，以及相關電位測量如何給出分析對象的酸鹼度值(pH)之工作原理？(15分)

三、(總分25分)傅立葉轉換(Fourier transform)在現代光譜儀的設計中普遍出現，也改善傳統光譜儀的缺點並提供了許多優勢，例如更高的靈敏度和分辨率(Jacquinot's advantage)，並且能更快地收集數據(Fellgett's advantage)，請說明相關優勢的來源以及與傳統光譜儀的差異？(15分)並說明傅立葉轉換是如何實現從時域(time domain)到頻域(frequency domain)的轉換，以及說明傅立葉轉換是否存在光譜波長範圍的使用限制？(10分)

四、(總分25分)t-測試、f-測試以及Q-測試常見於分析化學中數據統計分析，請說明各種測試分別適用何種數據分析情境，(15分)分析方法中偵測極限值(Detection limit)為選擇分析方法的重要標的，請說明如何定義偵測極限值並以統計觀點說明相關定義之來源。(10分)

（試題隨試卷繳回）